*Journal of Economic Theory and Econometrics*, Vol. 32, No. 4, Dec. 2021, 1–12

# Asymptotic Property of Least Squares Estimators for Explosive Autoregressive Models with a Drift*

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**Abstract** We study asymptotic inferences of the OLS estimator in the first order autoregressive model with an explosive root and a nonzero drift. Recent literatures focus on driftless model in dealing with explosive parameter in relation with financial bubbles detection. We consider an extension by allowing a non-zero drift, where the process behaves as a linear time trend during the non-bubble period, and it exhibits an exponential trend during the explosive era. Consistency of the least squares estimator and of the right-tailed coefficient-based Dickey-Fuller unit root test are shown in case of the presence of drift term.

**Keywords** Explosive Root, Drift, Consistency, Unit Root Test

**JEL Classification** C13, C22

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*The author is grateful to two anonymous referees for their insightful comments. Thanks also go to the managing editor, Noh-Sun Kwark.

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*Received June 24, 2021, Revised August 20, 2021, Accepted September 2, 2021*

## 1. INTRODUCTION

Non-stationary time series processes with possible explosive behaviors have recently drawn lots of attention in both theoretical and practical areas. Studies on autoregressive processes with explosive roots are pioneered by Phillips and Magdalinos (2007), where asymptotic inferences for explosive processes differ from unit root or near unit root asymptotics. Many subsequent researches in the context of explosive time series have mainly paid attention to detecting explosive movement of the series and estimating dates of emergence and collapse of asset price bubbles. Testing procedures for bubbles have been accumulated in the literature, which include recursive Dickey-Fuller(DF) or augmented DF tests by Phillips, Wu and Yu (2011; PWY hereafter), recursive DF test equipped with backward moving windows by Phillips, Shi and Yu (2015, PSY hereafter), reverse regression setup by Phillips and Shi (2018), CUSUM tests by Homm and Breitung (2012) and Breitung and Kruse (2010). In these testing mechanisms, the null process under non-bubble market periods becomes a pure unit root or a random walk process without a drift. Thus, stock prices are good examples in this regard.

While driftless models have many practical applications, some processes may contain a deterministic time trend along with periodically emerging and collapsing explosive components. In other words, the series of interest may behave as linear deterministic trends during the market period. The series turns to exhibit exponential trends as they enter into bubble period. As an example in this kind, house prices among other asset prices are often assumed to contain deterministic trends in some empirical studies (Gallin, 2006; Lee and Park, 2013). In this case, it is appropriate to perform right-tailed unit root tests where the true data generating processes consist of unit roots and a drift. Then, in relation to above-mentioned procedures, it is a question whether the LS estimator for the autoregressive parameter is still consistent, which is the main motivation of this work.

While test statistic is computed using subsamples, our case is understood as right-tailed version of the case 4 of unit root testing summarized in Hamilton (1994). Above-mentioned testing mechanisms fall on the right-tailed version of the case 2 of unit root tests. We analyze the asymptotic behavior of OLS estimator in the first-order explosive autoregressive model in the presence of a drift. It is found that the OLS estimator continues to be consistent in case of presence of drift term in the estimating model. Also, we consider the consistency of coefficient-based Dickey-Fuller test.

## 2. MAIN RESULTS

Consider a random walk with a drift,

$$y_t = \mu + y_{t-1} + \varepsilon_t, \tag{1}$$

where $\mu \neq 0$, $\varepsilon_t$ is $iid(0, \sigma^2)$. The partial sums of the errors satisfy $T^{-1/2}\sum_{j=1}^{t=[rT]} \varepsilon_j \Rightarrow B(r) = \sigma W(r)$, for $r \in [0,1]$, where $W$ is a standard Brownian motion and "$\Rightarrow$ "denote the weak convergence of the associated probability measure. The data generating process in (1) corresponds to the behavior of the series during the non-bubble period.

In recent context of testing for bubbles, zero or asymptotically negligible drift is often considered for the true data generating process. In our work, we instead explicitly consider non-negligible drift term, where, as widely known, the processes behave as a liner time trend, $y_t = y_0 + \mu t + \sum_{j=1}^{t} e_j$, for $y_0 = O_p(1)$. On the other hand, for explosive behavior of the process, we specify a mildly explosive autoregressive(AR) model with a drift(Phillips and Magdalinos, 2007),

$$\begin{aligned} y_t &= \mu + \delta_T y_{t-1} + \varepsilon_t, \\ \delta_T &= 1 + cT^{-\alpha}, \text{ for } c > 0 \text{ and } 0 < \alpha < 1, \end{aligned} \tag{2}$$

where the parameter $\alpha$ determines the speed of bubbles.

The above data generating process exists during the bubble period. In other words, the series behaves as unit root process with a drift during market (non-bubble) period as in (1), and by the onset of bubble period, it becomes explosive process as in (2). In other words, the process behaves as a linear trend under the market period, whereas it surges like an exponential trend under the bubble period. The sample paths of explosive process depend on the presence of drift, where literature mostly concentrate on driftless case. Besides, for the purpose of testing, the (1) and (2) can be understood as null and alternative process, respectively. A related work is Wang and Yu (2019), where some trend models with a possible bubble are studied.

In order to estimate the autoregressive parameter, a transformed regression model is considered (Hamilton, 1994),

$$y_t = \mu + \rho y_{t-1} + \theta t + \varepsilon_t = \mu^* + \rho x_{t-1} + \theta^* t + \varepsilon_t, \tag{3}$$

where $\mu^* = \mu(1-\rho)$, $x_{t-1} = y_{t-1} - \mu(t-1)$, and $\theta^* = \theta + \rho\mu$. Under the null hypothesis of unit root, $\mu^* = 0$, $\rho = 1$ and $\theta^* = \mu$, which implies the unit root process with a linear time trend.

Firstly, we focus on the estimator of $\delta_T$ in the regression model (3), which is typically computed using subsamples. To get some insights, we bring out the the standard unit root asymptotics(the case IV in Hamilton (1994)). Under the null hypothesis of $\rho = 1$ as well as $\theta = 0$, the asymptotic distribution of the full-sample OLS estimator are well developed by Phillips and Perron (1988). Suppose the estimation is computed over the subsamples from $\tau_1 = [r_1 T]$ to $\tau_2 = [r_2 T]$, for $0 < r_1 < r_2 \leq 1$, and $T$ is the sample size. Also, we set $\tau_w = [r_w T]$, where $r_w = r_2 - r_1$. Then the resulting limiting distribution of the subsample-based OLS estimator is given by

$$\tau_w(\hat{\rho} - 1) \sim_a \frac{\left[(4r_w^4 - 3r_w^2)\int_{r_1}^{r_2} W(s)dW(s) + A(r_1, r_2)\right]}{D(r_1, r_2)}, \tag{4}$$

where

$$A(r_1, r_2)$$
$$= (6r_w^3 - 4r_w^4)\int_{r_1}^{r_2} W(s)ds[W(r_2) - W(r_1)] + (6r_w^3 - 12r_w^2)\int_{r_1}^{r_2} sW(s)ds[W(r_2) - W(r_1)] +$$
$$12r_w^3 \int_{r_1}^{r_2} sW(s)ds \int_{r_1}^{r_2} W(s)ds - 6r_w^4 (\int_{r_1}^{r_2} W(s)ds)^2,$$

and

$$D(r_1, r_2)$$
$$= r_w^3 \left\{ r_w^2 \left[\int_{r_1}^{r_2} W^2(s)ds - 4(\int_{r_1}^{r_2} W(s)ds)^2\right] + 12 r_w \int_{r_1}^{r_2} W(s)ds \int_{r_1}^{r_2} sW(s)ds - 12(\int_{r_1}^{r_2} sW(s))^2 \right\}.$$

The notation " $\sim_a$ " denotes the asymptotic equivalence. If $r_1 = 0$ and $r_2 = 1$, then the limits of the LS estimators simply reduce to the full-sample version given in Phillips and Perron (1988)(Theorem 1) in the case of *iid* innovations.

Write the LS estimators as

$$\begin{pmatrix} \hat{\mu}^* \\ \hat{\rho} - 1 \\ \theta^* - \mu \end{pmatrix} = \begin{pmatrix} [r_w T] & \sum x_{j-1} & \sum j \\ \sum x_{j-1} & \sum x_{j-1}^2 & \sum j x_{j-1} \\ \sum j & \sum j y_{j-1} & \sum j^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \varepsilon_j \\ \sum x_{j-1}\varepsilon_j \\ \sum j \varepsilon_j \end{pmatrix} \tag{5}$$
$$= Q^{-1} R,$$

where the summation $\sum$ over $j$ runs from $\tau_1 = [r_1 T]$ to $\tau_2 = [r_2 T]$, $y_{t-1} = y_{t-1} - \rho(t-1)$. Also, let $\hat{\beta} = (\hat{\mu}^*, \hat{\rho} - 1, \hat{\theta}^* - \mu)'$.

As clarified in PWY and PSY, it is critical where the bubbles are located within the subsample for OLS estimation. Denote $\tau_e = [r_e T]$ and $\tau_f = [r_f T]$, for $0 \leq r_e < r_f \leq 1$ as the date of emergence and collapse of explosive bubbles, respectively. We denote $N_0 = [1, \tau_e), B = [\tau_e, \tau_f]$ and $N_1 = (\tau_f, \tau_2]$.Given this, we consider two different scenarios for recursive estimation, say, (i) $r_1 < r_e < r_2 < r_f$; the subsamples begin with random walks and end within an explosive era, and (ii) $r_e < r_1 < r_f < r_2$; the subsamples start in an explosive period, and finish in a non-explosive period. These two different cases are found to produce distinctive asymptotic behavior for the OLS estimators. Other cases produce identical results in the limit. For reference, the case of $r_1 < r_e < r_f < r_2$ turns out to be the same as the case of (ii). Also, to save a space, we restrict our attention to the case of a single bubble.

Below, we present the main result, which is the consistency of the OLS estimator for the autoregressive parameter.

**Theorem 1:** $(i)$ For $\tau_1 \in N_0$, $\tau_2 \in B$,

$$\hat{\rho} - \delta_T \sim_a q(r,c)B^{-1}(r_e)[\int_{r_1}^{r_e} B(r)dr \int_{r_1}^{r_e} rB(r)dr]\delta_T^{-(\tau_2-\tau_e)}T^{1-2\alpha}, \tag{6}$$

where $q(r,c)$ denotes a constant term consisting of $c, r_1, r_2$ and $r_e$.

$(ii)$ For $\tau_1 \in B$, $\tau_2 \in N_1$,

$$\hat{\rho} - \delta_T \sim_a -u(r,c)T^{-\alpha}, \tag{7}$$

where $u(r,c)$ is a constant term consisting of $c, r_1, r_2$ and $r_e$. Both $q(r,c)$ and $u(r,c)$ are detailed in the Appendix.

The part $(i)$ corresponds to the case that the series begins in a market period and ends in a bubble period. Conversely, the part $(ii)$ is associated with the case that the series begins in a bubble era and ends in a market era. The proof of Theorem 1 is given in the Appendix.

Remarks.

1. The above results show the consistency of OLS estimator for two difference scenarios, the part $(i)$ and $(ii)$. Note that the convergence rate of the LS estimator turns out to be much faster in the case $(i)$ than in the case $(ii)$, due to the presence of the term $\delta_T^{-(\tau_2-\tau_e)}$, which shows an exponential behavior. For the part $(ii)$, where subsamples start from an explosive era and encounter the collapse of bubbles, convergence rate depends on the parameter for explosion $\alpha$. The larger the value of $\alpha$, the faster the convergence of the estimator.

2. To confirm this theoretical inferences, we perform a small set of simulation studies. We consider the data generating process as $y_t = \mu + \delta_T y_{t-1} + e_t$, where $e_t$ is $iidN(0,1)$, $\mu = 0.1$, and $\delta_T = (1 + cT^{-\alpha})$. The sample size $T$ equals

to 200. For the part(i) of the Theorem 1, we set $c = 0$ for the first half the sample, and $c = 1$ for the remaining half, which implies that the bubble starts at the middle of the sample period and lasts until the end of the sample. We conduct 1,000 iterations, and calculate sample mean squared error(MSE) for $\rho$ as less than $T^{-10}$ for $\alpha = 0.1$, $0.3$, $2\times T^{-9}$ for $\alpha = 0.5$, and $2\times T^{-5}$ for $\alpha = 0.7$. Next, for the part(ii) of the Theorem, we set $c = 1$ for the first half the sample and $c = 0$ for the remaining half. It indicates that the bubbles lasts up to the half the sample and then collapse(i.e., the fraction of break date is 0.5). The sample mean squared error(MSE) is computed as 0.277, 0.062, 0.013 and 0.004, for $\alpha = 0.1$, 0.3, 0.5 and 0.7. As explained above, faster convergence in the part $(i)$ is then clearly confirmed. Also, the MSE values for each $\alpha$ decrease as sample size $T$ grows. The sample MSE of LS estimator in the case $(i)$ is found to go to zero at a significantly much faster rate than the case $(ii)$. It then confirms the theoretical conjecture. Simulation codes are available upon request.

The results in the Theorem 1 lead to the coefficient-based Dickey Fuller unit root testing, where the test statistic is denoted as $Z(\delta) = \tau_w(\hat{\rho} - 1)$. Below is the consistency of the $Z(\delta)$ test.

**Corollary 1:** $(i)$ For $\tau_1 \in N_0$, $\tau_2 \in B$, $Z(\delta) \to \infty$, and $(ii)$ For $\tau_1 \in B$, $\tau_2 \in N_1$, $Z(\delta) \to -\infty$.

Since $Z(\delta) = \tau_w(\hat{\rho} - \delta_T) + \tau_w(\delta_T - 1)$, when $\tau_1 \in N_0$, $\tau_2 \in B$, the second term, which is $cT^{1-\alpha}$, dominates and $Z(\delta) \to \infty$. When $\tau_1 \in B$, $\tau_2 \in N_1$, the test $Z(\delta)$ obtains consistency, if $u(r,c) > c$. Thus, consistency of coefficient-based DF test in PSY continues to hold in the presence of drift term. As we concentrate on the consistency of the LS estimator, the case of DF t-test is unexplored in this work.

## 3. CONCLUSION

We study asymptotic behaviors and consistency of OLS estimator for explosive parameter in autoregressive model with a nonzero drift. Our results are an extension of studies on explosive parameters in a driftless autoregressive model in the recent time series context. Proposed inferences fit for some assets in real world which follow linear deterministic trends during market period, and exhibit exponential trend during bubble period.

# APPENDIX

Let $\tau_1 = [r_1 T]$, $\tau_2 = [r_2 T]$ and $r_2 - r_1 = r_w > 0$, for $r_0 < r_1 < r_2 \leq 1$. Denote $\tau_e = [r_e T], \tau_f = [r_f T]$, where $\tau_e$ and $\tau_f$ denote the dates of emergence and collapse of explosive processes.

First, we summarize asymptotic results for $\sum_{\tau_1}^{\tau_2} x_{j-1}$ and $\sum_{\tau_1}^{\tau_2} x_{j-1}^2$ from PSY(Lemma A.1-A.4),

$$
\begin{aligned}
(i)\sum_{\tau_1}^{\tau_2} x_{j-1} &\sim_a \frac{T^{(1+2\alpha)/2}}{c}\delta_T^{(\tau_2-\tau_e)}B(r_e), \text{ when } \tau_1 \in N_0, \tau_2 \in B, \\
&\sim_a \frac{T^{(1+2\alpha)/2}}{c}\delta_T^{(\tau_f-\tau_e)}B(r_e), \text{ when } \tau_1 \in B,\ \tau_2 \in N_1, \\
(ii)\sum_{\tau_1}^{\tau_2} x_{j-1}^2 &\sim_a \frac{T^{1+\alpha}}{2c}\delta_T^{2(\tau_2-\tau_e)}B^2(r_e), \text{ when } \tau_1 \in N_0, \tau_2 \in B, \\
&\sim_a \frac{T^{1+\alpha}}{2c}\delta_T^{2(\tau_f-\tau_e)}B^2(r_e), \text{ when } \tau_1 \in B,\ \tau_2 \in N_1,
\end{aligned}
\tag{A.1}
$$

where $B(r) = \sigma W(r)$, with standard Brownian motion $W(r)$. Here, we do not separately consider the case of $\tau_1 \in N_0, \tau_2 \in N_2$ and $B \in [N_0, N_1]$, which is simply the sum of two cases above.

We provide the inference for $\sum_{\tau_1}^{\tau_2} jx_{j-1}$.

**Lemma 1.**

$$
\begin{aligned}
\sum_{\tau_1}^{\tau_2} jx_{j-1} &\sim_a \frac{(r_2-r_e)}{c}T^{(3+2\alpha)/2}\delta_T^{(\tau_2-\tau_e)}B(r_e), \text{ when } \tau_1 \in N_0,\ \tau_2 \in B, \\
&\sim_a \frac{(r_2-r_e)}{c}T^{(3+2\alpha)/2}\delta_T^{(\tau_f-\tau_e)}B(r_e), \text{ when } \tau_1 \in B,\ \tau_2 \in N_1, \\
&\sim_a \frac{(r_2-r_e)}{c}T^{(3+2\alpha)/2}\delta_T^{(\tau_f-\tau_e)}B(r_e), \text{ when } \tau_1 \in N_0,\ \tau_2 \in N_1.
\end{aligned}
$$

Firstly, for $\tau_1 \in N_0, \tau_2 \in B$, we get

$$
\sum_{\tau_1}^{\tau_2} jx_{j-1} = \sum_{\tau_1}^{\tau_e} jx_{j-1} + \sum_{\tau_e+1}^{\tau_2} jx_{j-1}. \tag{A.2}
$$

The first term is equal to

$$
\sum_{\tau_1}^{\tau_e} jx_{j-1} = T^{5/2}(r_e - r_1)\frac{1}{\tau_e - \tau_1}\sum_{\tau_1}^{\tau_e}\frac{j}{T}\frac{x_j}{T^{1/2}} \sim_a T^{5/2}(r_e - r_1)\int_{r_1}^{r_e} sB(s)ds, \tag{A.3}
$$

whereas, using the PSY(lemma A.1), the second term is written by

$$\begin{aligned}\textstyle\sum_{\tau_e+1}^{\tau_2} jx_{j-1} &= x_{\tau_e}\textstyle\sum_{\tau_e+1}^{\tau_2} j\delta_T^{j-1-\tau_e} \\ &= x_{\tau_e}\frac{(\tau_2-\tau_e)\delta_T^{(\tau_2-\tau_e)}}{(\delta_T-1)}(1+o_p(1)) \\ &= x_{\tau_e}\frac{(r_2-r_e)}{c}T^{\alpha+1}\delta_T^{(\tau_2-\tau_e)}(1+o_p(1)) \\ &\sim_a \frac{(r_2-r_e)}{c}T^{(3+2\alpha)/2}\delta_T^{(\tau_2-\tau_e)}B(r_e),\end{aligned} \tag{A.4}$$

where, in the second line, we get

$$\begin{aligned}\textstyle\sum_{\tau_e+1}^{\tau_2} j\delta_T^{j-1-\tau_e} &= \partial\left[(\delta_T^{(\tau_2-\tau_e+1)}-1)/(\delta_T-1)\right]/\partial\delta_T \\ &= \frac{(\tau_2-\tau_e+1)\delta_T^{(\tau_2-\tau_e)}}{(\delta_T-1)} - \frac{\delta_T^{(\tau_2-\tau_e+1)}-1}{(\delta_T-1)^2} = \frac{(\tau_2-\tau_e)\delta_T^{(\tau_2-\tau_e)}}{(\delta_T-1)}(1+o_p(1)).\end{aligned}$$

Thus, the second term dominates the first term in (A.2).

Next, for the cases of $\tau_1 \in B$ and $\tau_2 \in N_1$, and of $\tau_1 \in N_0$ and $\tau_2 \in N_2$, the results are derived by similar reasoning.

**Lemma 2.** We analyze the terms in the vector $R$. For $\tau_1 \in N_0$, $\tau_2 \in B$ and for $\mu \neq 0$,

$$\begin{aligned}(i)\ &\textstyle\sum_{\tau_1}^{\tau_2}(x_j-\mu-\delta_T x_{j-1}) \sim_a -c(r_e-r_1)T^{(3-2\alpha)/2}\int_{r_1}^{r_e}B(r)dr, \\ (ii)\ &\textstyle\sum_{\tau_1}^{\tau_2}x_{j-1}(x_j-\mu-\delta_T x_{j-1}) \sim_a -c(r_e-r_1)T^{(2-\alpha)}\int_{r_1}^{r_e}B^2(r)dr, \\ (iii)\ &\textstyle\sum_{\tau_1}^{\tau_2}j(x_j-\mu-\delta_T x_{j-1}) \sim_a -cT^{(5-2\alpha)/2}\int_{r_1}^{r_e}rB(r)dr.\end{aligned}$$

For $(i)$, we get

$$\begin{aligned}&\textstyle\sum_{\tau_1}^{\tau_2}(x_j-\mu-\delta_T x_{j-1}) \\ &= \textstyle\sum_{\tau_1}^{\tau_e-1}[x_j-\mu-x_{j-1}+(1-\delta_T)x_{j-1}] + \sum_{\tau_e}^{\tau_2}e_j \\ &\sim_a -c(r_e-r_1)T^{(3-2\alpha)/2}\int_{r_1}^{r_e}B(r)dr,\end{aligned} \tag{A.5}$$

where $1-\delta_T = -cT^{-\alpha}$. Next, write

$$\begin{aligned}&\textstyle\sum_{\tau_1}^{\tau_2}x_{j-1}(x_j-\mu-\delta_T x_{j-1}) \\ &= \textstyle\sum_{\tau_1}^{\tau_e-1}x_{j-1}[x_j-\mu-x_{j-1}+(1-\delta_T)x_{j-1}] + \sum_{\tau_e}^{\tau_2}e_j \\ &\sim_a \delta_T^{(\tau_2-\tau_e)}T^{(\alpha+1)/2}Z_cB(r_e) - c(r_e-r_1)T^{(2-\alpha)}\int_{r_1}^{r_e}B^2(r)dr \\ &\sim_a -c(r_e-r_1)T^{(2-\alpha)}\int_{r_1}^{r_e}B^2(r)dr,\end{aligned} \tag{A.6}$$

where the first term in third line $Z_c \sim N(0, \sigma^2/2c)$ comes from PSY(Lemma A.5). For the part $(iii)$, we obtain

$$\begin{aligned}
&\textstyle\sum_{\tau_1}^{\tau_2} j(x_j - \mu - \delta_T x_{j-1}) \qquad \text{(A.7)} \\
&= \textstyle\sum_{\tau_1}^{\tau_e - 1} j[x_j - \mu - x_{j-1} + (1-\delta_T)x_{j-1}] + \sum_{\tau_e}^{\tau_2} je_j \\
&= \textstyle\sum_{\tau_1}^{\tau_2} je_j - cT^{-\alpha} \sum_{\tau_1}^{\tau_e - 1} jx_{j-1} + \sum_{\tau_e}^{\tau_2} je_j \\
&\sim_a -cT^{(5-2\alpha)/2} \textstyle\int_{r_1}^{r_e} rB(r)dr.
\end{aligned}$$

**Lemma 3.** For $\tau_1 \in B$, $\tau_2 \in N_1$, and for $\mu \neq 0$,

$$(i)\ \textstyle\sum_{\tau_1}^{\tau_2} (x_j - \mu - \delta_T x_{j-1}) \sim_a -c\delta_T^{(\tau_f - \tau_e)} T^{(1-2\alpha)/2} B(r_e),$$

$$(ii)\ \textstyle\sum_{\tau_1}^{\tau_2} x_{j-1}(x_j - \mu - \delta_T x_{j-1}) \sim_a -c\delta_T^{2(\tau_f - \tau_e)} T^{1-\alpha} B^2(r_e),$$

$$(iii)\ \textstyle\sum_{\tau_1}^{\tau_2} j(x_j - \mu - \delta_T x_{j-1}) \sim_a -cr_f \delta_T^{(\tau_f - \tau_e)} T^{(3-2\alpha)/2} B(r_e).$$

For $(i)$, we use the results of $(i), (ii)$ in Lemma 4 and PSY(Lemma A.6) to get

$$\begin{aligned}
&\textstyle\sum_{\tau_1}^{\tau_2} (x_j - \mu - \delta_T x_{j-1}) \qquad \text{(A.8)} \\
&= \textstyle\sum_{\tau_1}^{\tau_f} e_j + (x_{\tau_f + 1} - \mu - \delta_T x_{\tau_f}) + \sum_{\tau_f + 2}^{\tau_2} [x_j - \mu - x_{j-1} + (1-\delta_T)x_{j-1}] \\
&= \textstyle\sum_{\tau_1}^{\tau_f} e_j + D_{1T} + \sum_{\tau_f + 1}^{\tau_2} e_j + D_{2T},
\end{aligned}$$

where

$$\begin{aligned}
D_{1T} &= x_{\tau_f + 1} - \mu - \delta_T x_{\tau_f} \qquad \text{(A.9)} \\
&= (x_{\tau_f} + \mu + e_{\tau_f + 1} - \mu - \delta_T x_{\tau_f}) \\
&= (1-\delta_T)(x_{\tau_f} - \mu\tau_f) + (1-\delta_T)\mu\tau_f + e_{\tau_f + 1} \\
&\sim_a -cT^{-\alpha} \delta_T^{(\tau_f - \tau_e)} T^{1/2} B(r_e) = -c\delta_T^{(\tau_f - \tau_e)} T^{(1-2\alpha)/2} B(r_e),
\end{aligned}$$

and $D_{2T} = O_p(T^{(3-2\alpha)/2})$. It follows that $D_{1T}$ dominates $D_{2T}$. Using the analogous reasoning, the part $(ii)$ is written as

$$\begin{aligned}
&\textstyle\sum_{\tau_1}^{\tau_2} x_{j-1}(x_j - \mu - \delta_T x_{j-1}) \qquad \text{(A.10)} \\
&= \textstyle\sum_{\tau_1}^{\tau_f} x_{j-1} e_j + x_{\tau_f}(x_{\tau_f + 1} - \mu - \delta_T x_{\tau_f}) + \sum_{\tau_f + 2}^{\tau_2} x_{j-1}[e_j + (1-\delta_T)x_{j-1}] \\
&= \textstyle\sum_{\tau_1}^{\tau_f} x_{j-1} e_j + D_{3T} + \sum_{\tau_f + 1}^{\tau_2} x_{j-1} e_j + D_{4T},
\end{aligned}$$

where

$$D_{3T} \sim_a -c\delta_T^{2(\tau_f-\tau_e)}T^{1-\alpha}B^2(r_e),$$

and $D_{4T} = O_p(T^{2-\alpha})$, which is a lower order term than $D_{3T}$. We continue to analyze the part $(iii)$ in a similar manner. Put

$$\begin{aligned} &\textstyle\sum_{\tau_1}^{\tau_2} j(x_j - \mu - \delta_T x_{j-1}) \\ &= \textstyle\sum_{\tau_1}^{\tau_f} je_j + (\tau_f+1)(x_{\tau_f+1} - \mu - \delta_T x_{\tau_f}) + \sum_{\tau_f+2}^{\tau_2} j(x_j - \mu - \delta_T x_{j-1}) \\ &= \textstyle\sum_{\tau_1}^{\tau_f} je_j + D_{5T} + D_{6T}, \end{aligned} \tag{A.11}$$

where

$$D_{5T} \sim_a -cr_f\delta_T^{(\tau_f-\tau_e)}T^{(3-2\alpha)/2}B(r_e),$$

and $D_{6T} = O_p(T^{(5-2\alpha)/2})$. Thus, $D_{5T}$ dominates $D_{6T}$.

Next, we study the limiting behavior of the matrix of regressors $Q$.

**Lemma 4.** Let $\Delta$ be the determinant of the matrix $Q$. From direct calculations, we get

$$\Delta \sim_a K(r,c)\delta_T^{2(\tau_2-\tau_e)}B^2(r_e)T^{5-\alpha}, \text{ for } \tau_1 \in N_0, \tau_2 \in B,$$

where $K(r,c)$ is a constant which consists of $r_1, r_2, r_e$ and $c$. We do not try to obtain the exact form of $K(r,c)$, as it has no impact on asymptotic behavior.

For reference, if the process is a pure random walk without an explosive component, then $\Delta = O_p(T^6)$ (cf: Phillips and Perron, 1988).

**Proof of Theorem 1.**

Denote $Q^{-1} = \Delta^{-1}G$, where $G$ is the adjugate of $Q$. Write the OLS estimator for the AR coefficient as

$$\hat{\rho} - \delta_T = \Delta^{-1}(G_{21}R_1 + G_{22}R_2 + G_{23}R_3) \tag{A.12}$$

where $G_{ij}$ denotes the $(i,j)$-th entry of the matrix $G$ and $R_i$ is the $i$-th entry of $R = (\sum_{\tau_1}^{\tau_2}(x_j - \mu - \delta_T x_{j-1}), \sum_{\tau_1}^{\tau_2} x_{j-1}(x_j - \mu - \delta_T x_{j-1}), \sum_{\tau_1}^{\tau_2} j(x_j - \mu - \delta_T x_{j-1}))'$.

First, consider the case $(i)$ when $\tau_1 \in N_0$, $\tau_2 \in B$. Direct computations together with Lemma 1 yield

$$G_{21}R_1 \sim_a n(r,c)\delta_T^{(\tau_2-\tau_e)}T^5B(r_e)\textstyle\int_{r_1}^{r_e} B(r)dr, \tag{A.13}$$

where $n(r,c)$ denotes a constant term consisting of $r_1, r_2, r_e$ and $c$. Next, we get $G_{22}R_2 = O_p(T^{6-\alpha})$, which is of lower order than $G_{21}R_1$, due to absence of $\delta_T^{(\tau_2-\tau_e)}$ term. We continue to obtain

$$G_{23}R_3 \sim_a p(r,c)\delta_T^{(\tau_2-\tau_e)}T^5B(r_e)\textstyle\int_{r_1}^{r_e} rB(r)dr, \tag{A.14}$$

where $p(r,c)$ denotes a constant term involving $r_1, r_2, r_e$ and $c$.

Thus, $G_{21}R_1$ and $G_{23}R_3$ have the same order of magnitude. Then, for $\tau_1 \in N_0$, $\tau_2 \in B$, we combine the results and the Lemma 4 to get

$$\hat{\rho} - \delta_T \sim_a q(r,c)B^{-1}(r_e)[\int_{r_1}^{r_e} B(r)dr \int_{r_1}^{r_e} rB(r)dr]\delta_T^{-(\tau_2-\tau_e)}T^{1-2\alpha}, \qquad \text{(A.15)}$$

where $q(r,c) = [n(r,c) + p(r,c)]K(r,c)$.

Next, consider the case $(ii)$ that $\tau_1 \in B$ and $\tau_2 \in N_1$. By similar reasoning as above, along with Lemma 5, we get

$$\begin{aligned} G_{21}R_1 &\sim_a -c \times n(r,c)B^2(r_e)\delta_T^{2(\tau_f-\tau_e)}T^4, \qquad \text{(A.16)} \\ G_{22}R_2 &\sim_a -s(r,c)B^2(r_e)\delta_T^{2(\tau_f-\tau_e)}T^{5-\alpha}, \\ G_{23}R_3 &\sim_a -cr_f p(r,c)B^2(r_e)\delta_T^{2(\tau_f-\tau_e)}T^4, \end{aligned}$$

where $m(r,c) = cn(r,c)$, and $s(r,c)$ is a constant term consisting of $r_1, r_2, r_e$ and $c$.

It follows that the term $G_{22}R_2$ is a dominant term. Thus, the limiting property in the case of $\tau_1 \in B$ and $\tau_2 \in N_1$ is given by

$$\hat{\rho} - \delta_T \sim_a -u(r,c)T^{-\alpha}, \qquad \text{(A.17)}$$

where $u(r,c) = s(r,c)K(r,c)$.

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